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A bound on the plurigenera of projective surfaces

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Abstract

We exhibit a sharp Castelnuovo bound for the ith plurigenus of a smooth minimal surface of general type and of given degree d in the projective space $\mathbf{P}^r$, and classify the surfaces attaining the bound, at least when $d \gg r$. We give similar results for surfaces not necessarily minimal or of general type, but only for $i \gg r$ (however, in the case $r \leq 8$, we give a complete classification, i.e., for any $i \geq 1$). In certain cases (only for $r \geq 12$) the surfaces with maximal plurigenus are not Castelnuovo surfaces, i.e., surfaces with maximal geometric genus. © 2001 Elsevier Science B.V. All rights reserved.

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Halphen [7] and Castelnuovo [2] proved a sharp upper bound for the genus of projective curves of given degree. Harris [8] extended this result to the geometric genus of projective varieties of arbitrary dimension. In this paper we exhibit similar results for the plurigenera of projective surfaces.

In order to formulate our main results, fix integers r and d and let $\mathscr{V}(r,d)$ be the set of all smooth, irreducible, projective and nondegenerate surfaces V of degree d in the projective space $\mathbf{P}^r$. Let $\mathscr{V}'(r,d)$ be the set of all minimal surfaces $V \in \mathscr{V}(r,d)$ of general type. For any integer $i \geq 1$ put

$$P(r,d,i) = \sup\{p_i(V): V \in \mathscr{V}'(r,d)\} \tag{0.1}$$

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and

$$P'(r,d,i) = \sup\{p_i(V) : V \in \mathcal{V}'(r,d)\}, \tag{0.2}$$

where $p_i(V)$ denotes the ith plurigenus of V. In the case $i=1$ and for $d \gg r$, Harris [8] proved that

$$P(r,d,1) = \binom{m}{3}(r-2) + \varepsilon\binom{m}{2}, \tag{0.3}$$

where m and ε are defined by dividing

$$d-1 = m(r-2)+\varepsilon, \quad 0 \le \varepsilon < r-2, \tag{0.4}$$

and he classified the surfaces with maximal geometric genus $P(r,d,1)$, the so-called *Castelnuovo surfaces*. In this paper we compute $P'(r,d,i)$ in the case $i \ge 2$ and classify the surfaces $V \in \mathcal{V}'(r,d)$ which achieve the bound, at least when $d \gg r$. Then we prove similar results for $P(r,d,i)$, but only for $i \gg r$ (however, in the case $r \le 8$, we give a complete classification, i.e., for any $i \ge 2$). More precisely, first we will show the following theorem concerning minimal surfaces of general type.

Theorem A. *Let $V \subset \mathbf{P}^r$ $(r \ge 4)$ be a smooth, irreducible and nondegenerate projective surface of degree d. Denote by $p_i(V)$ the ith plurigenus of V. Assume V is minimal and of general type* (*i.e.*, $V \in \mathcal{V}'(r,d)$), $d > 2r^2-8$ *and* $i \ge 2$. *With the same notation as in* (0.4) *define*

$$\varepsilon_0(r) = (5r-8)/6,$$

$$\lambda(r,d,i) = m\left[\binom{i}{2}(5r-6\varepsilon-8)+r-2-\varepsilon\right] - \binom{i}{2}(4r-3\varepsilon-11)$$

and

$$k(r,d) = (m-2)[d(m-2)+2(m+1)(\varepsilon-1)]. \tag{0.5}$$

We have

(i) *if* $(r,\varepsilon) \ne (6,3)$ *and either* $0 < \varepsilon < \varepsilon_0(r)$, *or* $\varepsilon \ge \varepsilon_0(r)$ *and* $\lambda(r,d,i) \ge 0$, *then* $p_i(V) \le 1 + P(r,d,1) + \binom{i}{2}k(r,d)$ (*see* (0.3));
(ii) *if* $\varepsilon \ge \varepsilon_0(r)$ *and* $\lambda(r,d,i) < 0$, *then* $p_i(V) \le 1 + P(r,d,1) + \binom{i}{2}k(r,d) - \lambda(r,d,i)$;
(iii) *if* $\varepsilon = 0$, *then* $p_i(V) \le 1 + P(r,d,1) + \binom{i}{2}(k(r,d)+1)$;
(iv) *if* $(r,\varepsilon) = (6,3)$, *then* $p_i(V) \le 1 + P(6,d,1) + \binom{i}{2}(k(6,d)+d/4)$.

All previous bounds are sharp. Therefore, in the appropriate ranges, they are equal to $P'(r,d,i)$ (*see* (0.2)). *Moreover*:

- *In case* (i) *we have* $p_i(V) = P'(r,d,i)$ *if and only if either V is residual to* $r-3-\varepsilon$ *planes in the complete intersection of a rational normal scroll of dimension* 3 *whose singular locus has dimension* <1, *with a hypersurface of degree* $m+1$, *or, only in the case* $\varepsilon \ge \varepsilon_0(r)$ *and* $\lambda(r,d,i) = 0$, *V is residual to* $2r-5-\varepsilon$ *planes in the complete intersection of a rational normal scroll of dimension* 3 *whose singular locus has* *dimension* <1, *with a hypersurface of degree* $m+2$, *or*, *only in the case* $r=4$, V *is a complete intersection of a quadric with a hypersurface of degree* $m+1$.

- In case (ii) we have $p_i(V)=P'(r,d,i)$ *if and only if* V *is residual to* $2r-5-\varepsilon$ *planes in the complete intersection of a rational normal scroll of dimension* 3 *whose singular locus has dimension* <1, *with a hypersurface of degree* $m+2$.
- In case (iii) *we have* $p_i(V)=P'(r,d,i)$ *if and only if* V *is residual to* $r-3$ *planes in the complete intersection of a cone over a smooth rational normal scroll surface having a line directrix* (*or*, *only for* $r=4$, *of a cone over a conic with vertex a line*), *with a hypersurface of degree* $m+1$.
- *In case* (iv) *we have* $p_i(V)=P'(6,d,i)$ *if and only if* V *is the complete intersection of a cone in* $\mathbf{P}^6$ *over the Veronese surface with a hypersurface of degree* $m+1$ *not containing the vertex of the cone.*

In cases (iii) and (iv) the surfaces with maximal ith plurigenus ($i\geq 2$) are Castelnuovo's (but the converse does not hold). This is not true in cases (i) (partly) and (ii). The new surfaces appearing in our classification are not arithmetically Cohen–Macaulay (on the contrary Castelnuovo surfaces are). However they are surfaces with irregularity $q=0$, i.e., their geometric genus is equal to the arithmetic one (see (1.5)).

Taking into account a vanishing theorem for pluricanonical divisors on a minimal surface of general type (see (1.1) below), we will see that Theorem A is a consequence of the classification of Castelnuovo surfaces [8] and of surfaces with maximal self-intersection of the canonical bundle [4] (this accounts for the hypothesis $d>2r^2-8$ in the statement of Theorem A).

Next, as an application of the previous Theorem A, we prove the following theorem concerning surfaces not necessarily minimal or of general type.

Theorem B. *Let* $V\subset\mathbf{P}^r$ ($r\geq 4$) *be a smooth*, *irreducible and nondegenerate projective surface of degree* d (*i.e.*, $V\in\mathscr{V}(r,d)$). *Assume* $d>2r^2-8$ *and* $i\geq 2$ *for* $4\leq r\leq 8$, *and* $d>4r^2$ *and* $i\geq r/3$ *for* $r\geq 9$. *Then*, *in the appropriate ranges*, *all bounds appearing in the statement of Theorem* A *hold for the ith plurigenus* $p_i(V)$ *of* V. *These bounds are sharp and*, *therefore*, *they are equal to* $P(r,d,i)$ (see (0.1)). *With our numerical hypotheses*, *the classification of the surfaces* $V\in\mathscr{V}(r,d)$ *with maximal plurigenus* $P(r,d,i)$ *is the same as in Theorem* A. *In particular*, *one has* $P(r,d,i)=P'(r,d,i)$.

Notice that for $r\geq 9$ (and $d>4r^2$) the classification given in Theorem B is not complete because of the hypothesis $i\geq r/3$. But recall that any smooth surface can be embedded in $\mathbf{P}^5$, where Theorem B provides complete results for $d>42$.

By using Castelnuovo–Halphen's theory, we reduce the proof of Theorem B to an analysis of the surfaces $V\in\mathscr{V}(r,d)$ lying on threefolds of minimal degree $r-2$ (see Propositions 2 and 3 and Corollary 4 below). Then we accomplish this analysis by using Castelnuovo–Halphen's theory again and reducing the proof to Theorem A.

The assumption $i \geq r/3$ (only for $r \geq 9$) and $d \gg r$ is essential in our proof of Theorem B. We need it for using Theorem A, Castelnuovo–Halphen's theory (Proposition 3 and Corollary 4) and for several numerical estimates in the proof of Proposition 3 and at the end of the proof of Theorem B (see (0.8)). However this hypothesis is certainly not the best possible. It is only of the simplest form we were able to conceive (see also Remark 5).

In Remark 6, we point out that Harris' approach [8] in proving the bound $P(r,d,1)$ (see (0.3)) for the geometric genus $p_1(V)$ of a surface V does not work in the study of plurigenera $p_i(V)$ with $i \geq 2$. When $r > 8$ we do not know whether Theorem B holds also for $2 \leq i < r/3$.

We work over the complex number field $\mathbf{C}$ and we use standard notation of Algebraic Geometry.

We begin by showing Theorem A. To this end we need the following lemma. In the sequel $\mathscr{L}(D)$ means the invertible sheaf associated to a divisor D.

Lemma 1. *For any degree $d > 2r^2 - 8$, the surfaces appearing in the statement of Theorem A exist, are minimal, of general type and, for any $i \geq 2$, their ith plurigenus achieves the corresponding numerical upper bound.*

Proof. For the existence of such surfaces in $\mathscr{V}(r,d)$ we refer to [4, Proposition (2.3), Remark (2.4)].

In case (i), let $V \in \mathscr{V}(r,d)$ be a surface residual to $r-3-\varepsilon$ planes in the complete intersection of a rational normal scroll of dimension 3 whose singular locus has dimension < 1, with a hypersurface of degree $m+1$. By [8] we know that V is a Castelnuovo surface, it is minimal, of general type and the arithmetic genus $p_a(V)$ of V is equal to the geometric genus $p_1(V) = P(r,d,1)$ (see (0.3)). Since V is minimal and of general type, then by Barth et al. [1, Theorem I.7.2, Proposition VII.5.5] we have the following vanishing (K_V denotes the canonical divisor of V):

$$h^1(V, \mathscr{L}(iK_V)) = h^2(V, \mathscr{L}(iK_V)) = 0 \quad \text{for any } i \geq 2. \tag{1.1}$$

By Riemann–Roch theorem, we deduce for any $i \geq 2$

$$p_i(V) = h^0(V, \mathscr{L}(iK_V)) = 1 + p_a(V) + \binom{i}{2} K_V^2 = 1 + P(r,d,1) + \binom{i}{2} k(r,d).$$

In case (ii), let V be a surface residual to $2r-5-\varepsilon$ planes in the complete intersection of a rational normal scroll T of dimension 3 whose singular locus has dimension < 1, with a hypersurface of degree $m+2$. Suppose for a moment that T is smooth. Since V is linearly equivalent on T to $(m+2)H + (\varepsilon + 1 - 2(r-2))W$ (here H (resp. W) denotes the generic hyperplane section (resp. a plane of the ruling) of T) then the canonical divisor K_V is the restriction of $(m-1)H + (\varepsilon + 1 - r)W$ to V. By Di Gennaro [4, Proposition (2.3) and its proof] we know that K_V is nef, hence V is minimal. From the natural exact sequence

$$0 \to \mathscr{L}(K_T) \to \mathscr{L}(K_T + V) \to \mathscr{O}_V \otimes \mathscr{L}(K_T + V) \to 0, \tag{1.2}$$

we get

$$p_1(V) = h^0(T, \mathscr{L}(K_T + V)) = (r-2)\binom{m+1}{3} + (\varepsilon - r + 2)\binom{m+1}{2}$$

(see [6, Corollary 2.2.9 and its proof]). In particular $p_1(V) > 0$. Since $K_V^2 > 0$ we deduce that V is of general type. Now we are going to compute $p_i(V)$. By (1.1) and [4, (2.2.2) and (0.4)] it follows that

$$\begin{aligned} p_2(V) &= 1 + p_a(V) + K_V^2 \leq 1 + p_1(V) + K_V^2 \\ &= 1 + (r-2)\binom{m+1}{3} + (\varepsilon - r + 2)\binom{m+1}{2} + k(r,d,1), \end{aligned} \tag{1.3}$$

where for any integer a we define

$$k(r,d,a) = (m-2+a)[d(m-2+a) + 2(m+1+a)(\varepsilon - 1 - a(r-2))] \tag{1.4}$$

(notice that $k(r,d) = k(r,d,0)$ (see (0.5))). On the other hand, from sequence (1.2) tensored with $\mathscr{L}(K_T + V)$ we deduce

$$\begin{aligned} p_2(V) &\geq h^0(T, \mathscr{L}(2K_T + 2V)) - h^0(T, \mathscr{L}(2K_T + V)) \\ &= 1 + (r-2)\binom{m+1}{3} + (\varepsilon - r + 2)\binom{m+1}{2} + k(r,d,1). \end{aligned}$$

From (1.3) we get

$$p_a(V) = p_1(V) \tag{1.5}$$

and therefore, for any $i \geq 2$, we have

$$\begin{aligned} p_i(V) &= 1 + p_a(V) + \binom{i}{2} K_V^2 \\ &= 1 + (r-2)\binom{m+1}{3} + (\varepsilon - r + 2)\binom{m+1}{2} + \binom{i}{2} k(r,d,1) \\ &= 1 + P(r,d,1) + \binom{i}{2} k(r,d) - \lambda(r,d,i). \end{aligned}$$

Arguing in a similar way and taking into account [4], one examines the case T is a cone, the case of a complete intersection, and the remaining cases (ii)–(iv) of the statement of Theorem A. This concludes the proof of Lemma 1. □

We are in position to prove Theorem A. To this purpose, consider a surface $V \in \mathscr{V}'(r,d)$. Since V is a minimal smooth surface of general type, then we may use (1.1) and by Riemann–Roch theorem, we get for any $i \geq 2$

$$p_i(V) = 1 + p_a(V)\binom{i}{2} K_V^2 \leq 1 + p_g(V) + \binom{i}{2} K_V^2.$$

Therefore, taking into account previous Lemma 1, our Theorem A is a consequence of the classification of projective surfaces with maximal geometric genus [8] and of surfaces with maximal self-intersection of the canonical bundle [4]. This concludes the proof of Theorem A. $\square$

Now we are going to prove Theorem B. We need some preliminaries (i.e., Propositions 2 and 3 and Corollary 4 below) which enable us to reduce the proof to an analysis of the surfaces lying on threefolds of minimal degree.

Proposition 2. *Let $V \subset \mathbf{P}^r$ be a smooth, irreducible, projective surface of degree d. Denote by H the generic hyperplane section of V and by g the genus of H. Let E be any divisor on V. Then one has*

$$h^0(V, \mathscr{L}(E)) \leq [(2E \cdot H + d + 4)^2 + (2E \cdot H + d - 4g)^2]/16d.$$

Proof. We may assume E is effective. Let j_0 be the maximal nonnegative integer j such that $h^0(V, \mathscr{L}(E - jH)) > 0$, and denote by j_1 the maximal nonnegative integer $j \leq j_0$ such that $h^1(H, \mathscr{L}(E - jH) \otimes \mathcal{O}_H) = 0$ (if there is no such integer put $j_1 = -1$). Assume for a moment $j_1 \geq 0$. Since $h^1(H, \mathscr{L}(E - jH) \otimes \mathcal{O}_H) = 0$ for any $0 \leq j \leq j_1$ then, from the natural exact sequence

$$0 \to \mathscr{L}(E - H) \to \mathscr{L}(E) \to \mathcal{O}_H \otimes \mathscr{L}(E) \to 0$$

and using Riemann–Roch theorem on H, we deduce

$$h^0(V, \mathscr{L}(E)) \leq h^0(V, \mathscr{L}(E - H)) + 1 - g + E \cdot H.$$

Using the same inequality for $E - H$, we get (if $j_1 > 0$)

$$h^0(V, \mathscr{L}(E)) \leq h^0(V, \mathscr{L}(E - 2H)) + 2(1 + E \cdot H) - d - 2g.$$

Continuing in this fashion, we obtain the following inequality:

$$h^0(V, \mathscr{L}(E)) \leq h^0(V, \mathscr{L}(E - (j_1 + 1)H)) + (j_1 + 1)(1 + E \cdot H - j_1 d/2 - g) \tag{2.1}$$

which holds true also when $j_1 = -1$. Now, by using Clifford's theorem on H, we can estimate $h^0(V, \mathscr{L}(E - jH))$ for $j_1 < j \leq j_0$. Then a similar computation as before shows that

$$h^0(V, \mathscr{L}(E - (j_1 + 1)H)) \leq (j_0 - j_1)[(E - (j_1 + 1)H) \cdot H/2 - (j_0 - j_1 - 1)d/4 + 1].$$

By (2.1) we deduce

$$h^0(V, \mathscr{L}(E)) \leq \varphi(j_0) + \psi(j_1), \tag{2.2}$$

where for any rational number q, we put

$$\varphi(q) = (1 + q)(E \cdot H/2 - dq/4 + 1) \quad \text{and} \quad \psi(q) = (1 + q)(E \cdot H/2 - dq/4 - g).$$

Divide $E \cdot H = \alpha d + \beta$, $0 \leq \beta < d$ and notice that since $(E - (\alpha + 1)H) \cdot H = \beta - d < 0$ then $j_0 \leq \alpha$. It follows that $\varphi(j_0) \leq \varphi(\alpha)$ because in the range $0 \leq q \leq \alpha$ (and $q \in \mathbf{Z}$)

the function $\varphi(q)$ is increasing. Hence, we have

$$\begin{aligned}\varphi(j_0) &\le \varphi(\alpha)\\ &= [(E\cdot H)^2 + (d+4)E\cdot H + (\beta+4)(d-\beta)]/4d\\ &\le (2E\cdot H + d + 4)^2/16d. \qquad (2.3)\end{aligned}$$

On the other hand, the function $\psi(q)$ takes its maximum when $q=(E\cdot H - d/2 - 2g)/d$. It follows that

$$\begin{aligned}\psi(j_1) &\le \psi((E\cdot H - d/2 - 2g)/d)\\ &= (2E\cdot H + d - 4g)^2/16d. \qquad (2.4)\end{aligned}$$

Our claim follows by (2.2)–(2.4). $\square$

We need the previous proposition and Castelnuovo–Halphen's theory [3,5] for proving Proposition 3. We recall that the functions $P(r,d,1)$ and $k(r,d)$ which appear in its statement are defined in (0.3) and (0.5).

Proposition 3. *Let $V\subset \mathbf{P}^r$ $(r\ge 4)$ be a smooth, irreducible and nondegenerate projective surface of degree d, not contained in any variety of dimension 3 and degree $< r-1$. Denote by $p_i(V)$ the ith plurigenus of V. Assume $d > 2r^2-8$ and $i\ge 2$ for $4\le r\le 8$, and $d>4r^2$ and $i\ge r/3$ for $r\ge 9$. Then one has*

$$p_i(V) < 1 + P(r,d,1) + \binom{i}{2} k(r,d). \qquad (3.1)$$

Proof. By Proposition 2 (put $E = iK_V$ and use the adjunction formula) we have

$$p_i(V) = h^0(V, \mathscr{L}(iK_V)) \le \gamma(g,d,i)/16d, \qquad (3.2)$$

where g denotes the genus of the generic hyperplane section $H\subset \mathbf{P}^{r-1}$ of V and

$$\gamma(g,d,i) = [2i(2g-2-d)+d+4]^2 + [2i(2g-2-d)+d-4g]^2.$$

By the numerical assumptions on d and by the lifting theorem [3, Theorem 0.2], we have that H is not contained in any surface of $\mathbf{P}^{r-1}$ of degree $<r-1$. By Castelnuovo–Halphen's theory [3, Theorems 3.13 and 3.15] we get

$$g \le (r-1)\binom{\mu}{2} + \mu(v+1) + 1,$$

where μ and v are defined by dividing $d-1=\mu(r-1)+v$, $0\le v< r-1$. It follows that

$$g \le [d^2 + d(r-3) + r]/2(r-1).$$

Assume for a moment $g\ge d$. In this range the function $\gamma(g,d,i)$ is increasing with respect to g. Therefore we may insert the previous bound for g into (3.2) obtaining with elementary calculations

$$p_i(V) \le d^2[d(2i^2-2i+1) - 4i^2 + 4i + r - 5]/4(r-1)^2$$

which holds also for $g < d$ (use (3.2) again). On the other hand, by using (0.3)–(0.5), with elementary calculations again one obtains

$$1 + P(r,d,1) + \binom{i}{2} k(r,d)$$
$$> d^2[d(3i^2 - 3i + 1) - 3(r-1)(4i^2 - 4i + 1)]/6(r-2)^2. \tag{3.3}$$

Summing up we get

$$1 + P(r,d,1) + \binom{i}{2} k(r,d) - p_i(V)$$
$$> d^2[(i^2 - i)(6d(2r-3) - 24(r-1)^3 + 12(r-2)^2) + d(2(r-1)^2$$
$$-3(r-2)^2) - 6(r-1)^3 - 3(r-2)^2(r-5)]/12(r-1)^2(r-2)^2$$
$$= d^3[6(i^2 - i)(2r-3) - r^2 + 8r - 10]/12(r-1)^2(r-2)^2 + O(d^2). \tag{3.4}$$

The coefficient of d^3 in (3.4) is > 0 for any $i \geq 2$ when $4 \leq r \leq 8$ and for any $i \geq r/3$ when $r \geq 9$. It follows that, in the appropriate range for i, (3.1) holds for any $d \gg r$ and $d \gg i$. A more accurate but elementary analysis of (3.4) shows that (3.1) holds for $d > 2r^2 - 8$ when $4 \leq r \leq 7$, and for $d > 4r^2$ when $r \geq 9$. Similar computations show our claim in the case $r = 8$. $\square$

As a consequence of Lemma 1 and Proposition 3 we get the following.

Corollary 4. *Let $V \subset \mathbf{P}^r$ $(r \geq 4)$ be a smooth, irreducible and nondegenerate projective surface of degree d. Denote by $p_i(V)$ the ith plurigenus of V. Assume $d > 2r^2 - 8$ and $i \geq 2$ for $4 \leq r \leq 8$, and $d > 4r^2$ and $i \geq r/3$ for $r \geq 9$. If $p_i(V) = P(r,d,i)$ (see (0.1)) then V is contained in a 3-dimensional subvariety of $\mathbf{P}^r$ of minimal degree $r - 2$.*

Remark 5. Inequality (3.4) shows that the numerical hypotheses of Proposition 3 (and therefore of Corollary 4) are not the best possible. For instance, one can see that our claim holds for any $i \geq 1/2 + \sqrt{(r-3)/12}$ when $d > 4r^3$ and $r \geq 9$. Also notice that for $i = 2$ and $r \geq 31$ the leading coefficient in (3.4) is < 0. Hence the assumption $i \gg r$ is necessary in our approach in proving Proposition 3.

We are in position to prove Theorem B. To this purpose fix a surface $V \in \mathscr{V}(r,d)$. By Corollary 4 we may assume V lying on a threefold $T \subset \mathbf{P}^r$ of minimal degree $r - 2$. T is either a rational normal scroll or a smooth quadric in $\mathbf{P}^4$ or a cone over a Veronese surface in $\mathbf{P}^6$.

First, we examine the case of T is a smooth threefold rational normal scroll in $\mathbf{P}^r$ (a fortiori $r \geq 5$). In this case there exists a unique integer $a = a(V,T)$ such that V is linearly equivalent to $(m + 1 + a)H + (\varepsilon + 1 - (a+1)(r-2))W$ on T, where H (resp. W) denotes the generic hyperplane section (resp. a plane of the ruling) of

T [4,6,8]. Notice that $a \geq -m$. The canonical divisor K_V of V is the restriction of $(m-2+a)H + (\varepsilon - 1 - a(r-2))W$ to V. We deduce

$$K_V^2 = k(r,d,a) \tag{0.6}$$

(see (1.4)) and that the genus of the generic hyperplane section of V is

$$g(a) = (m+a)[(m-a-1)(r-2) + 2\varepsilon]/2 \tag{0.7}$$

from which we get $a \leq m$ because $g(a) \geq 0$. Therefore, we have

$$-m \leq a \leq m.$$

When $a = -m$ we have $g(-m) = 0$. Hence V is rational and then all its plurigenera vanish. In particular, $p_i(V) = 0 < 1 + P(r,d,1) + \binom{i}{2}k(r,d)$ for all $i \geq 2$ and $d > 2r^2 - 8$.

When $-m+1 \leq a \leq -m+2$, taking into account (0.7), one can estimate the plurigenera of V in a similar way as in the proof of Proposition 3 (cf. with (3.2)) and prove that $p_i(V) < 1 + P(r,d,1) + \binom{i}{2}k(r,d)$ for all $i \geq 2$ and $d > 2r^2 - 8$.

If $-m+3 \leq a \leq (m+\varepsilon-3)/(r-3)$ then V is minimal [4, Proposition 2.3 and its proof]. A direct computation shows that in this case $K_V^2 > 0$, i.e. $k(r,d,a) > 0$ (see (0.6)). Hence V is either rational or of general type. Thus we can reduce the proof of Theorem B to Theorem A.

If $(m+\varepsilon-3)/(r-3) < a \leq m$, then by (3.2) and (0.7) we have $p_i(V) \leq \gamma(g(a),d,i)/16d$. Since in our range $g(a)$ is decreasing one sees that $g(a) \leq g((m+\varepsilon-3)/(r-3))$. Then, as in the proof of Proposition 3, we get

$$\begin{aligned} p_i(V) &\leq \gamma(g((m+\varepsilon-3)/(r-3)),d,i)/8d \\ &= (2i^2-2i+1)(r-4)^2d^3/4(r-3)^4 + \mathrm{O}(d^2). \end{aligned}$$

By (3.3) we obtain

$$\begin{aligned} &12(r-2)^2(r-3)^4\left[1 + P(r,d,1) + \binom{i}{2}k(r,d) - p_i(V)\right] \\ &\quad \geq [6(i^2-i)(2r^2-12r+17) - r^4 + 8r^3 - 18r^2 + 26]d^3 + \mathrm{O}(d^2). \end{aligned} \tag{0.8}$$

The coefficient of d^3 in (0.8) is >0 for any $i \geq 2$ when $4 \leq r \leq 8$ and for any $i \geq r/3$ when $r \geq 9$. Therefore, in the appropriate range for i, one has $p_i(V) < 1 + P(r,d,1) + \binom{i}{2}k(r,d)$ for any $d \gg r$ and $d \gg i$. A direct computation shows that the previous inequality holds for $d > 2r^2 - 8$ when $4 \leq r \leq 8$, and for $d > 4r^2$ when $r \geq 9$. This concludes our analysis when T is a smooth threefold rational normal scroll.

Now we turn to the remaining cases. Let V be a smooth surface of degree $d > 2r^2 - 8$ lying on a singular threefold rational normal scroll $T \subset \mathbf{P}^r$ ($r \geq 4$), or on a smooth quadric in $\mathbf{P}^4$, or on a cone over a Veronese surface in $\mathbf{P}^6$. By Di Gennaro [4, Lemma (2.1), Propositions (2.2), (2.3) and their proof] one sees that V is minimal and of general type (in fact it is Castelnuovo's), except when T is a cone over a smooth surface rational normal scroll in $\mathbf{P}^{r-1}$, $\varepsilon = 0$ and V is isomorphic to its strict transform

$\tilde{V}$ on the minimal rational resolution $\mathbf{F}$ of T. In this case, for some integer a, we have $\tilde{V} \sim (m+1+a)\tilde{H} + (1-(a+1)(r-2))\tilde{W}$ and then we may argue as in the case of T is smooth. This concludes the proof of Theorem B.

Remark 6. We point out that Harris' approach [8] in proving the bound $P(r,d,1)$ (see (0.3)) for the geometric genus $p_1(V)$ of a surface $V \in \mathscr{V}(r,d)$ does not work in the study of plurigenera $p_i(V)$ with $i \geq 2$, in the following sense.

In the case of geometric genus, by using the Poincaré residue sequence

$$0 \to \Omega_V^2(-j) \to \Omega_V^2(-j+1) \to \Omega_H^1(-j) \to 0,$$

first, one proves that for any $V \in \mathscr{V}(r,d)$ one has

$$p_1(V) \leq \sum_{j \geq 1} (j-1)(d - c_V^1(j)), \tag{6.1}$$

where $c_V^1(j) = h^0(H, \mathscr{L}(j\Gamma)) - h^0(H, \mathscr{L}((j-1)\Gamma))$ (here H (resp. Γ) denotes the generic hyperplane section of V (resp. H)). Next one shows the crucial step, i.e., that there exists a *sharp lower bound* for the function $c_V^1(j)$, given by the function $c_S^1(j) = \min\{d, j(r-2)+1\}$, where $S \in \mathscr{V}(r,d)$ is any Castelnuovo surface. This means that for any $V \in \mathscr{V}(r,d)$, for any Castelnuovo surface $S \in \mathscr{V}(r,d)$ and for any $j \geq 0$ one has

$$c_V^1(j) \geq \min\{d, j(r-2)+1\} = c_S^1(j) \quad \text{and} \quad p_1(S) = \sum_{j \geq 1} (j-1)(d - c_S^1(j)). \tag{6.2}$$

Finally by (6.1) one obtains the bound

$$p_1(V) \leq \sum_{j \geq 1} (j-1)(d - c_S^1(j)) = p_1(S) = P(r,d,1).$$

In the case of plurigenera inequality (6.1) becomes

$$p_i(V) \leq \sum_{j \geq 1} (j-1)(d - c_V^i(j)), \tag{6.3}$$

where $c_V^i(j) = h^0(H, \mathscr{L}(j\Gamma + (i-1)(\Gamma - K_H))) - h^0(H, \mathscr{L}((j-1)\Gamma + (i-1)(\Gamma - K_H)))$. Now, to simplify our discussion, assume $\varepsilon = 1$. In this case, by our Theorem B, we know that the surfaces S with maximal plurigenera are Castelnuovo's. Therefore, for any $V \in \mathscr{V}(r,d)$, one has

$$p_i(V) \leq p_i(S) = P(r,d,i) = \sum_{j \geq 1} (j-1)(d - c_S^i(j)) \tag{6.4}$$

(when $\varepsilon = 1$ then $c_S^i(j) = 0$ for $j < (i-1)(m-2)$ and $c_S^i(j) = c_S^1(j - (i-1)(m-2))$ for $j \geq (i-1)(m-2)$ (see (6.2))). But there is no hope to generalize the inequality appearing in (6.2), i.e., there is no sharp lower bound for the function $c_V^i(j)$ when $i \geq 2$. In fact, if such a lower bound existed then, by (6.3) and (6.4), it would have to be equal to $c_S^i(j)$. This is impossible for if we take a surface V contained in a smooth threefold rational normal scroll $T \subset \mathbf{P}^r$ linearly equivalent on T to $mH + 2W$, then we have

$$c_V^i((m-2)(i-1)) = 0 < c_S^i((m-2)(i-1)) = 1.$$

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